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Path integration of a two-time quadratic action

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Abstract. Path integration of a general two-time quadratic action characterising memory effects is performed within the framework of Feynman's polygonal path approach. Explicit evaluation of the propogator in exact analytical form is further carried out for the specific kernel used by Feynman in the polaron problem.

1. Introduction

In his path integral theory for polarons, Feynman (1955) introduced for the first time a non-local quadratic action of the form

$$S[x(t)]=\tfrac{1}{2}m\int_0^T \dot{x}^2\,\mathrm{d}t-\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\,G(t,s)((x(t)-x(s))^2 \qquad (1.1)$$

where $G(t, s)$ is (without loss of generality) a symmetric function of t and s. As a simple way of characterising memory effects, this action has, since then, been used in many physical problems. Apart from its use by a number of authors in the polaron problem (Feynman 1955, Krivoglaz and Pekar 1957, Osaka 1958, Hellwarth and Platzman 1962, Thornber 1971, Sa Yakanit 1979), this action was also considered by Bezak (1970) for treating an electron gas in a random potential. It has also been exploited in the calculation of the density of electronic states in disordered systems (Edwards and Guylayev 1964, Samathiyakanit 1974, Gross 1977, Sa Yakanit 1979) and in discussing the propagation of waves in random media (Chow 1972, Dashen 1979).

Most of the above applications use either a simple form of the memory kernel (e.g. $G(s, t)$ = constant, Bezak 1970) or an approximate solution of the classical equation of motion. An exact propagator for action (1.1) using Bezak's kernel was first obtained by Papadopoulos (1974) and subsequently by others (Maheshwari 1975, Khandekar *et al* 1981, Dhara *et al* 1982) employing different techniques of path integration. Explicit evaluation of the propagator for an arbitrary kernel $G(s, t)$ has received attention only very recently (Adamowski and Gerlach 1982, Dhara *et al* 1982).

In this paper, we carry out the path integration of the general action (1.1) within the framework of Feynman's polygonal approach (Feynman and Hibbs 1965) and in the spirit of some of our previous work (Khandekar and Lawande 1975, 1979). We obtain an exact analytical form for the propagator which has the form of a free particle

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propagator with an 'effective mass' apart from the normalisation factor. We show that both the exponent and normalisation factor are related to the solutions of certain integrodifferential equations.

We present in § 2 the derivation of the propagator. The formulation developed in § 2 is subsequently applied in § 3 to obtain explicitly an exact propagator for a specific form of the kernel $G(s, t)$. Section 4 summarises the essential results.

2. Derivation of the propagator

2.1. Basic formulation

In the Feynman path-integral formulation of quantum mechanics, the propagator is defined as

$$K(x, T; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x} \exp\{(\mathrm{i}/\hbar)S[x(t)]\}\mathscr{D}[x(t)] \tag{2.1}$$

where the symbol $\mathscr{D}[x(t)]$ implies that integrations are performed over all possible paths from $x(0) = x_0$ to $x(T) = x$. Now, using the polygonal paths approach (Feynman and Hibbs 1965), we write equation (2.1) as

$$K(x, T; x_0, 0) = \lim_{N\to\infty} K_N(x, T; x_0, 0) \tag{2.2a}$$

with K_N defined by

$$K_N = A_N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} \mathrm{d}x_j \exp[(\mathrm{i}/\hbar)S_N]. \tag{2.2b}$$

Here S_N is the discretised form of the action defined over the partition of the time interval $[0, T]$ into N subintervals each of length ε. Thus $x_j = x(t_j)$, $x_0 = x(0)$, $x_N = x$ and $t_j - t_{j-1} = T/N$. A_N is the usual normalisation factor

$$A_N = (m/2\pi\mathrm{i}\hbar\varepsilon)^{N/2}. \tag{2.3}$$

The discretised form of the action of (1.1) reads

$$S_N = \frac{m}{2\varepsilon}\sum_{j=1}^{N}(x_j - x_{j-1})^2 - \varepsilon^2\sum_{j=1}^{N}\sum_{k=1}^{N} G_{jk}(x_j - x_k)^2,$$
$$G_{jk} = G(t_j, t_k), \tag{2.4}$$

and K_N takes the form

$$K_N(x_N, T; x_0, 0) = \left(\frac{\alpha}{\pi}\right)^{N/2} \exp[-\alpha(x_0^2 + x_N^2)] \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{X} \exp\{-\alpha[(\boldsymbol{X}, P\boldsymbol{X}) - 2(\boldsymbol{X}, \boldsymbol{Y})]\}. \tag{2.5}$$

Here P is an $(N-1)$-dimensional square matrix with the following structure

$$P_{ij} = P_{ji}, \qquad P_{jj} = 2(1 + g_{jj} - \Omega_j^2), \tag{2.6}$$
$$P_{j,j+1} = 2g_{j,j+1} - 1, \qquad P_{ij} = 2g_{ij} \qquad (i \neq j, j \pm 1)$$

where

$$g_{ij}=\frac{2\varepsilon^3}{m}G_{ij}, \qquad \Omega_j^2=\sum_{i=1}^{N} g_{ij}; \tag{2.7}$$

and $\boldsymbol{X}$ and $\boldsymbol{Y}$, are column vectors having $(N-1)$ components:

$$\boldsymbol{X}=\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_{N-1}\end{pmatrix} \qquad \boldsymbol{Y}=\begin{pmatrix} x_0\\ 0\\ 0\\ \vdots\\ 0\\ x_N\end{pmatrix} \tag{2.8}$$

and $\alpha=(m/2\mathrm{i}\hbar\varepsilon)$. The symbols $\mathrm{d}\boldsymbol{X}=\Pi_{j=1}^{N-1}\,\mathrm{d}x_j$. The gaussian integral in (2.5) may be easily evaluated employing the general formula

$$\int_{-\infty}^{\infty}\exp[-(\boldsymbol{X},A\boldsymbol{X})+a(\boldsymbol{X},\boldsymbol{Y})]\,\mathrm{d}\boldsymbol{X}$$
$$=(\pi^{n-1}/\det(A))^{1/2}\exp[\tfrac{1}{4}a^2(\boldsymbol{Y},A^{-1}\boldsymbol{Y})] \tag{2.9}$$

and we arrive at

$$K_N=(\alpha/\pi\det(P))^{1/2}\exp\{\alpha[(\boldsymbol{Y},P^{-1}\boldsymbol{Y})-x_0^2-x_N^2]\}. \tag{2.10}$$

It is now clear that the explicit form of the propagator is obtained if we are able to obtain the exponent

$$p=\alpha[(\boldsymbol{Y},P^{-1}\boldsymbol{Y})-x_0^2-x_N^2] \tag{2.11}$$

and the normalisation factor

$$q=(\alpha/\pi\det(P))^{1/2} \tag{2.12}$$

in the limit $\varepsilon\to 0$, $N\to\infty$ with $N\varepsilon=T$.

2.2. *Exponent p*

We introduce a new vector $\boldsymbol{U}$ such that

$$P\boldsymbol{U}=\boldsymbol{Y}. \tag{2.13}$$

Written in component form (2.13) reads

$$2(1+g_{jj}-\Omega_j^2)u_j+(2g_{j,j+1}-1)u_{j+1}$$
$$+(2g_{j,j-1}-1)u_{j-1}+2\sum_{k=1}^{N-1} g_{jk}u_k=0, \qquad N-1\geqslant j\geqslant 1, \tag{2.14}$$

with the endpoint values defined as

$$u_0=x_0, \qquad u_N=x_N. \tag{2.15}$$

The next step involves converting the difference equation (2.14) into a differential form by going over to the $\varepsilon\to 0$ limit. For this purpose, we substitute in (2.14) the explicit expressions (2.7) for g_{ij} and Ω_j^2, divide throughout by ε^2 and take the limit as

$\varepsilon \to 0$. The resulting integrodifferential equation is given by

$$\tfrac{1}{4}m\ddot{u} + \int_0^T G(t, s)(u(t) - u(s))\, \mathrm{d}s = 0, \qquad \left(\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}t}\right) \tag{2.16}$$

along with the end-point conditions

$$u(0) = x_0, \qquad u(T) = x_N. \tag{2.17}$$

Note that (2.16) is just the classical equation of motion obtained by varying $S[x(t)]$ in (1.1). Next on integrating (2.16) over the time interval $[0, T]$ we obtain

$$\dot{u}(0) = \dot{u}(T). \tag{2.18}$$

Further, the general solution of (2.16), satisfying the condition (2.17), has the form

$$u(t) = c + [(x_N - x_0)/(v_N - v_0)]v(t) \tag{2.19a}$$

where

$$c = (x_0 v_N - x_N v_0)/(v_N - v_0) \tag{2.19b}$$

and $v(t)$ is a non-constant solution of (2.16). We now use these results for evaluating the exponent p. Employing equations (2.8) and (2.13) we have

$$\begin{aligned}(\boldsymbol{Y}, P^{-1}\boldsymbol{Y}) &= (\boldsymbol{Y}, \boldsymbol{U}) = x_0 u_1 + x_N u_{N-1} \\ &= x_0[u(0) + \varepsilon\dot{u}(0) + \mathrm{O}(\varepsilon^2)] + x_N[u(T) - \varepsilon\dot{u}(T) + \mathrm{O}(\varepsilon^2)] \\ &= x_0^2 + x_N^2 + \varepsilon[x_0\dot{u}(0) - x_N\dot{u}(T)] + \mathrm{O}(\varepsilon^2).\end{aligned} \tag{2.20}$$

Making use of equations (2.18)–(2.20) we obtain

$$\lim_{\varepsilon \to 0} p = (\mathrm{i}m/2\hbar)[(x - x_0)^2/(v(T) - v(0))]\dot{v}(T). \tag{2.21}$$

The expression on the right is just equal to $(\mathrm{i}/\hbar)S_{\mathrm{cl}}$, S_{cl} being the action computed along the classical path from x_0 to x.

2.3. *Normalisation factor q*

The first step in obtaining the normalisation factor is to decompose the matrix P as

$$P = L + V$$

where L and V are $(N-1)$-dimensional square matrices. This decomposition is not unique. One may choose L and V in any manner provided L^{-1} exists. In particular, one may choose $L = I$ (unit matrix). For subsequent discussion we choose L and V such that

$$L_{ij} = L_{ji}, \qquad L_{j,j+1} = -1, \qquad L_{jj} = 2(1 - \Omega_j^2), \qquad L_{ij} = 0 \qquad (i \neq j, j \pm 1), \tag{2.22}$$

$$V_{ij} = 2g_{ij} = V_{ji}, \tag{2.23}$$

with the decomposition

$$\det(P) = (\det(L)(\det(I + L^{-1}V)). \tag{2.24}$$

2.3.1. Evaluation of $\det(I+L^{-1}V)$. We now proceed to obtain an explicit expression for $\det(I+L^{-1}V)$ by employing as a general result

$$\det(I-\lambda K)=\exp\left(+\int_{\lambda}^{0}\mathrm{d}\mu\ \mathrm{Tr}\ R(\mu)\right), \tag{2.25}$$

where K is a finite-dimensional square matrix and the matrix $R(\mu)$ satisfies the equation

$$R(\mu)=K+\mu KR. \tag{2.26}$$

In our case $\lambda=-1$, $K=L^{-1}V$ and it is more convenient to set up an equation for $\tilde{R}=R/\varepsilon$

$$\tilde{R}=\varepsilon^{-1}L^{-1}V+\mu L^{-1}V\tilde{R}$$

i.e.

$$L\tilde{R}=\varepsilon^{-1}V+\mu V\tilde{R}. \tag{2.27}$$

Employing equations (2.22) and (2.23) we can express (2.27) in the component form:

$$(2-2\Omega_j^2)\tilde{R}_{j,k}-\tilde{R}_{j-1,k}-\tilde{R}_{j+1,k}$$
$$=\varepsilon^{-1}V_{j,k}+2\mu\sum_{l=1}^{N-1}g_{jl}\tilde{R}_{lk},\qquad 1\leqslant j,\ k\leqslant N-1, \tag{2.28}$$

subject to the conditions

$$\tilde{R}_{0k}=\tilde{R}_{Nk}=0\qquad \forall k. \tag{2.29}$$

Next, substituting in equation (2.28) the expressions (2.7) for Ω_j and g_{jl}, rearranging and dividing throughout by ε^2 we take the limit $\varepsilon\to 0$. Equation (2.28) then goes over to the following integrodifferential form

$$\tfrac{1}{4}m(\partial^2/\partial t^2)\tilde{R}(t,s)+\Gamma(t)\tilde{R}(t,s)=G(t,s)+\mu\int_0^T G(t,t'')\tilde{R}(t'',s)\,\mathrm{d}t'', \tag{2.30}$$

with

$$\tilde{R}(0,s)=\tilde{R}(T,s)=0\qquad \forall s\in[0,T], \tag{2.31}$$

$$\Gamma(t)=\int_0^T G(t,s)\,\mathrm{d}s. \tag{2.32}$$

Now according to equation (2.25)

$$\det(I+L^{-1}V)=\exp\left(\int_{-1}^{0}\mathrm{d}\mu\ \mathrm{Tr}\ R(\mu)\right)=\exp\left(\int_{-1}^{0}\mathrm{d}\mu\sum_{j=1}^{N-1}\varepsilon\tilde{R}_{jj}\right)$$

and hence

$$Q_1=\lim_{\varepsilon\to\infty}\det(I+L^{-1}V)=\exp\left(\int_0^T\mathrm{d}t\int_{-1}^{0}\mathrm{d}\mu\,\tilde{R}(t,t,\mu)\right).$$

2.3.2. Evaluation of $\det(L)$. From the definition of the matrix L in (2.22), it is easy to see that Δ_k, the kth minor of $\det(L)$, satisfies the recursion relation

$$\Delta_k=2(1-\Omega_k^2)\Delta_{k-1}-\Delta_{k-2},\qquad k\geqslant 1, \tag{2.34}$$

with

$$\Delta_0 = 1, \qquad \Delta_{-1} = 0. \tag{2.35}$$

Writing $\psi_{k+1} = \varepsilon \Delta_k$ one immediately obtains from (2.34) and (2.35) the equations

$$\psi_{k+1} - 2\psi_k + \psi_{k-1} + 2\Omega_k^2 \psi_k = 0, \qquad k \geq 2,$$

$$\psi_0 = 0, \qquad \psi_1 = \varepsilon.$$

Dividing throughout by ε^2 and taking the limit $\varepsilon \to 0$ one arrives at the differential equation

$$\tfrac{1}{4} m \ddot{\psi} + \Gamma(t) \psi = 0 \tag{2.36}$$

with the initial conditions

$$\psi(0) = 0, \qquad \dot{\psi}(0) = 1, \tag{2.37}$$

and $\Gamma(t)$ defined as in (2.32).

The solution of the differential equation (2.36) satisfying the initial conditions (2.37) reads

$$\psi(t) = [\eta(t)\xi(0) - \xi(t)\eta(0)]/d \tag{2.38}$$

where ξ and η are two linearly independent solutions of (2.36) and

$$d = [\dot{\eta}(0)\xi(0) - \dot{\xi}(0)\eta(0)]. \tag{2.39}$$

Finally, since $\det(L) = \Delta_{N-1} = \psi_N/\varepsilon$ we arrive at the result

$$Q_2 = \lim_{\varepsilon \to 0} (\varepsilon \det(L)) = [\eta(T)\xi(0) - \xi(T)\eta(0)/d. \tag{2.40}$$

Employing equations (2.12), (2.33) and (2.40) we then obtain

$$\lim_{\varepsilon \to 0} q = \lim_{\varepsilon \to 0} (m/2\pi i\hbar)^{1/2} (\varepsilon \det(L))^{-1/2} (\det(I + L^{-1} V))^{-1/2}$$

$$= (m/2\pi i\hbar Q_1 Q_2)^{1/2}, \tag{2.41}$$

where Q_1 and Q_2 are as in equations (2.33) and (2.40) respectively.

2.4. *The propagator*

According to (2.2*a*), (2.10)–(2.12), and (2.41), the required propagator has the closed analytical form

$$K(x, T; x_0, 0) = (m/2\pi i\hbar Q_1 Q_2)^{1/2} \exp((i/\hbar) S_{cl}) \tag{2.42}$$

where Q_1 and Q_2 are defined in (2.33) and (2.40) while S_{cl} is the action evaluated along the classical path:

$$S_{cl} = \tfrac{1}{2} m \dot{x}(T)(x - x_0) = \tfrac{1}{2} m (x - x_0)^2 \dot{v}(T)/(v(T) - v(0)). \tag{2.43}$$

Note that the propagator has a form similar to that for a free particle with 'effective mass'

$$m^* = mT\dot{v}(T)/(v(T) - v(0)) \tag{2.44}$$

while the normalisation factor contains an additional term $(m^* Q_1 Q_2/mT)^{-1/2}$ apart from the free particle normalisation factor $(m^*/2\pi i\hbar T)^{1/2}$. Further, noting that $m^* = T(\partial^2 S_{cl}/\partial x \partial x_0)$ one finds that the propagator of (2.42) is essentially given by the

Van-Vleck–Pauli formula

$$C_f[(2\pi i\hbar)^{-1}|\partial^2 S_{cl}/\partial x \partial x_0|]^{1/2} \exp[(i/\hbar]S_{cl}] \tag{2.45}$$

apart from the correction factor $C_f = (m^* Q_1 Q_2/mT)^{-1/2}$.

Explicit evaluation of the propagator depends upon the knowledge of the kernel $G(s, t)$ and whether or not the basic equations (2.16), (2.30) and (2.36) yield analytical solutions for a given $G(s, t)$. Nevertheless our general formulation opens the door for trying out a variety of kernels $G(s, t)$ depending on the physical problem considered. In § 3 we apply our method to treat a kernel $G(s, t)$ which incidentally has been employed by Feynman in the polaron problem.

3. Explicit evaluation for a particular kernel

We now apply the theory developed in § 2 to obtain, in an exact closed form, the propagator for the case when the kernel $G(t, s)$ has the form

$$G(t, s) = \tfrac{1}{4} m\Omega^2 \omega^2 \phi(t, s) \tag{3.1}$$

where

$$\phi(t, s) = \cos[\omega(\tfrac{1}{2}T - |t-s|)]/2\omega \sin(\tfrac{1}{2}\omega T). \tag{3.2}$$

It will be useful to consider some of the properties of $\phi(t, s)$. Firstly $\phi(t, s)$ is symmetric in t and s and is normalised such that

$$\int_0^T \phi(t, s) \mathrm{d}s = 1/\omega^2. \tag{3.3}$$

Secondly, $\phi(t, s)$ satisfies the differential equation

$$(\mathrm{D}^2 + \omega^2)\phi(t, s) = \delta(t-s), \qquad \mathrm{D} = \partial/\partial t. \tag{3.4}$$

Thirdly $\phi(t, s)$ and its derivative with respect to t, $\mathrm{D}\phi(t, s)$, obey the end-point conditions

$$\begin{aligned} &\phi(0, s) = \phi(T, s) = \cos[\omega(\tfrac{1}{2}T - s)]/2\omega \sin(\tfrac{1}{2}\omega T) \\ &\mathrm{D}\phi(0, s) = \mathrm{D}\phi(T, s) = -\omega \sin[\omega(\tfrac{1}{2}T - s)]/2\omega \sin(\tfrac{1}{2}\omega T). \end{aligned} \tag{3.5}$$

Lastly, we have the following easily proved identity

$$\begin{aligned} \int_0^T \phi(t, s) f(s)\, \mathrm{d}s \\ &= (\mu^2 - \omega^2)^{-1}[-f(t) + \mathrm{D}\phi(0, t)(f(T) - f(0)) \\ &\quad - \phi(0, t)(\mathrm{D}f(T) - \mathrm{D}f(0)] \end{aligned} \tag{3.6}$$

where $f(t)$ is a solution of the differential equation

$$(\mathrm{D}^2 + \mu^2) f(t) = 0. \tag{3.7}$$

The evaluation of the propagator proceeds in three stages. We first consider the exponent which requires the solution of (2.16). This equation now takes the form

$$(\mathrm{D}^2 + \Omega^2) u = \Omega\omega^2 \int_0^T \phi(t, s) u(s)\, \mathrm{d}s. \tag{3.8}$$

Applying the operator $(\mathrm{D}^2+\omega^2)$ on both sides and employing the property (3.4), we arrive at the ordinary differential equation

$$(\mathrm{D}^2+\nu^2)\mathrm{D}^2 u=0$$

whose solution is readily obtained as

$$u=F+v(t) \tag{3.9}$$

where

$$\nu=(\Omega^2+\omega^2)^{1/2}, \qquad v(t)=A\sin(\nu t)+B\cos(\nu t)+Ct \tag{3.10}$$

where A, B, C and F are constants. The idea is to use the non-constant part of the solution, namely $v(t)$, and determine the coefficients A, B and C so that $v(t)$ also satisfies the original equation (3.8). Substituting $v(t)$ from (3.10) in (3.8) and using the property (3.6) to evaluate the right-hand side we obtain two consistency conditions:

$$A(1-\cos(\nu t))+B\sin(\nu t)=0, \tag{3.11}$$

$$A\omega^2\sin(\nu t)-\omega^2 B(1-\cos(\nu t))+C\Omega^2 T=0. \tag{3.12}$$

These conditions then determine A and C in terms of B. As a result we obtain

$$v(t)=B\left[\frac{\cos(\nu(T-t)]-\cos(\nu t)}{\cos(\nu T)-1}+\frac{2\omega^2 t}{\Omega^2 T}\right] \tag{3.13}$$

as the required non-constant solution of (3.8). Using this solution we obtain the quantities $\dot{v}(T)$, $v(0)$ and $v(T)$ required to compute S_{cl} of equation (2.43). Thus

$$S_{\mathrm{cl}}=\tfrac{1}{2}(m\Omega^2/\nu^2)[(\omega^2/\Omega^2 T)+\tfrac{1}{2}\nu\cot(\tfrac{1}{2}\nu T)](x-x_0)^2. \tag{3.14}$$

Next, we consider the evaluation of the normalisation factor. The part corresponding to $\det(L)$ involves the solution of (2.36) which for the present case reads as

$$\mathrm{d}^2\psi/\mathrm{d}t^2+\Omega^2\psi=0, \qquad \psi(0)=0, \qquad \dot{\psi}(0)=1. \tag{3.15}$$

Two linearly independent solutions of equation (3.15) are

$$\xi(t)=\sin\Omega t, \qquad \eta(t)=\cos\Omega t. \tag{3.16}$$

Hence the part of the normalisation factor, namely Q_2 of (2.40), is obtained as

$$Q_2=\sin(\Omega t)/\Omega. \tag{3.17}$$

Lastly, we consider the part Q_1 of the normalisation factor corresponding to $\det(I+L^{-1}V)$. For this purpose we consider equations (2.30), (2.31) which take the form

$$(D^2+\Omega^2)\tilde{R}(t,s)=\Omega^2\omega^2\phi(t,s)+\mu\Omega^2\omega^2\int_0^T\phi(t,r)\tilde{R}(r,s)\,\mathrm{d}r,$$

$$\tilde{R}(0,s)=\tilde{R}(T,s)=0. \tag{3.18}$$

We try to arrive at the solution of this equation in the same manner as that employed earlier for equation (3.8). Operating on both sides of (3.18) by $(\mathrm{D}^2+\omega^2)$ and using the property (3.4), we obtain the linear differential equation

$$(\mathrm{D}^2+\Omega_1^2)(\mathrm{D}^2+\Omega_2^2)\tilde{R}(t,s)=\Omega^2\omega^2\delta(t-s), \tag{3.19}$$

where

$$\Omega_{1,2}^2=\tfrac{1}{2}(\Omega^2+\omega^2)\mp[\tfrac{1}{4}(\Omega^2-\omega^2)^2+\mu\Omega^2\omega^2]^{1/2}. \tag{3.20}$$

The solution of (3.19) is then given by

$$\tilde{R}(t, s) = [\Omega^2\omega^2/(\Omega_2^2 - \Omega_1^2)][\tilde{R}_1(t, s) - \tilde{R}_2(t, s)], \tag{3.21}$$

where $\tilde{R}_j$ ($j = 1, 2$) are the solutions of the equation

$$(\mathrm{D}^2 + \Omega_j^2)\tilde{R}_j(t, s) = \delta(t - s), \tag{3.22}$$

along with the conditions

$$\tilde{R}_1(0, s) = \tilde{R}_2(0, s), \qquad \tilde{R}_1(T, s) = \tilde{R}_2(T, s). \tag{3.23}$$

The next step involves substitution of $\tilde{R}(t, s)$ of (3.21) into the integrodifferential equation (3.18) and determining the boundary conditions satisfied by $\tilde{R}_k(t, s)$ so that $\tilde{R}(t, s)$ is indeed a solution of (3.18). This can be carried out by making use of the identity

$$\int_0^T \phi(s, t)\tilde{R}_j(t, r)\, \mathrm{d}t = \theta_j^{-2}[\phi(s, r) - R_j(s, r) - A_j], \tag{3.24}$$

where

$$\theta_j^2 = \Omega_j^2 - \omega^2, \tag{3.25}$$

$$A_j = \phi(0, s)[\mathrm{D}\tilde{R}_j(T, r) - \mathrm{D}\tilde{R}_j(0, r)] - \mathrm{D}\phi(0, s)[\tilde{R}_j(T, r) - \tilde{R}_j(0, r)] \tag{3.26}$$

for simplifying the right-hand side of (3.18). After some algebra, one obtains the additional boundary conditions to be imposed on $\tilde{R}_j(t, s)$. These are

$$\tilde{R}_j(0, s) = \tilde{R}_j(T, s), \tag{3.27}$$

$$\theta_2^2[\mathrm{D}\tilde{R}_1(T, s) - \mathrm{D}\tilde{R}_1(0, s)] = \theta_2^2[\mathrm{D}\tilde{R}_2(T, s) - \mathrm{D}\tilde{R}_2(0, s)]. \tag{3.28}$$

Next, the general solution $\tilde{R}_j$, $j = 1, 2$, equation (3.22) satisfying the condition (3.27) has the form

$$\tilde{R}_j(t, s) = P_j(t, s) + a\chi_j(t, s) \tag{3.29}$$

where

$$P_j(t, s) = -\frac{\sin(\Omega_j t)\sin[\Omega_j(T - s)]}{\Omega_j \sin(\Omega_j T)}, \qquad t \leqslant s, \tag{3.30}$$

$$= -\frac{\sin(\Omega_j s)\sin[\Omega_j(T - t)]}{\Omega_j \sin(\Omega_j T)} \qquad t \geqslant s, \tag{3.21}$$

$$\chi_j = \frac{\sin(\Omega_j t) + \sin[\Omega_j(T - t)]}{\sin(\Omega_j T)}. \tag{3.32}$$

Now, applying the second condition we determine the remaining constant a:

$$a = \tfrac{1}{2}(\theta_2^2\chi_2(s) - \theta_1^2\chi_1(s))/(\theta_1^2 f_2 - \theta_2^2 f_1), \tag{3.33}$$

where

$$f_j = \Omega_j[\cos(\Omega_j T) - 1]/\sin(\Omega_j T). \tag{3.34}$$

Hence the complete solution of (3.18) for $\tilde{R}(t, s)$ reads as

$$\tilde{R}(t, s) = \frac{\Omega^2\omega^2}{\Omega_2^2 - \Omega_1^2}\left(P_1(t, s) - P_2(t, s) - \frac{[\theta_2^2\chi_1(s) - \theta_1^2\chi_2(s)][\chi_1(t) - \chi_2(t)]}{2(\theta_1^2 f_2 - \theta_2^2 f_1)}\right). \tag{3.35}$$

Next, in order to evaluate Tr $\tilde{R}$ we need the results

$$\int_0^T P_j(t,t)\,\mathrm{d}t = \tfrac{1}{2}\left(\frac{T\cos(\Omega_j T)}{\Omega_j \sin(\Omega_j T)} - \frac{1}{\Omega_j^2}\right), \tag{3.36}$$

$$\int_0^T \chi_j^2\,\mathrm{d}t = -2\,\mathrm{d}f_j/\mathrm{d}\Omega_j^2. \tag{3.37}$$

With the help (3.35)–(3.37), the expression for Tr $\tilde{R}$ takes the form

$$\begin{aligned}\mathrm{Tr}\,\tilde{R}(\mu) &= \int_0^T \tilde{R}(t,t,\mu)\,\mathrm{d}t \\ &= \frac{\Omega^2\omega^2}{(\Omega_2^2-\Omega_1^2)}\left[\left(\frac{T\cot(\Omega_1 T)}{2\Omega_1} - \frac{1}{2\Omega_1^2}\right) - \left(\frac{T\cot(\Omega_2 T)}{2\Omega_2} - \frac{1}{2\Omega_2^2}\right)\right] \\ &\quad - \frac{\left[\theta_2^2\dfrac{\mathrm{d}f_1}{\mathrm{d}\Omega_1^2} + \theta_1^2\dfrac{\mathrm{d}f_2}{\mathrm{d}\Omega_2^2} + \dfrac{(\theta_1^2+\theta_2^2)(f_1-f_2)}{(\Omega_2^2-\Omega_1^2)}\right]}{(\theta_1^2 f_2 - \theta_2^2 f_1)}.\end{aligned} \tag{3.38}$$

We have now to evaluate the quantity $\int_{-1}^{0}\mathrm{d}\mu\ \mathrm{Tr}\,\tilde{R}(\mu)$. For this purpose, it is convenient to make a transformation from μ to λ,

$$\lambda = (\lambda_0^2 + \mu\omega^2\Omega^2)^{1/2} = \Omega_2^2 - \Omega_1^2, \tag{3.39}$$

$$\lambda_0 = \tfrac{1}{2}(\Omega^2-\omega^2), \qquad \lambda_1 = \tfrac{1}{2}(\Omega^2+\omega^2), \qquad \Omega_{1,2} = (\lambda_1 \mp \lambda)^{1/2},$$

so that

$$\begin{aligned}\int_{-1}^{0}\mathrm{d}\mu\ \mathrm{Tr}\,\tilde{R}(\mu) &= \tfrac{1}{2}\int_{\lambda_1}^{\lambda_0}\mathrm{d}\lambda\left(\frac{\cot[T(\lambda_1-\lambda)^{1/2}]}{(\lambda_1-\lambda)^{1/2}} - \frac{\cot[T(\lambda_1+\lambda)^{1/2}]}{(\lambda_1+\lambda)^{1/2}}\right. \\ &\quad + \frac{1}{(\lambda_1+\lambda)^{1/2}} - \frac{1}{(\lambda_1-\lambda)^{1/2}} \\ &\quad \left. + \frac{(\lambda_0+\lambda)\,\mathrm{d}f_1/\mathrm{d}\lambda - (\lambda_0-\lambda)\,\mathrm{d}f_2/\mathrm{d}\lambda - (\lambda_0/\lambda)(f_1-f_2)}{(\lambda_0-\lambda)f_2 - (\lambda_0+\lambda)f_1}\right).\end{aligned} \tag{3.40}$$

The integration over λ is easily performed yielding the result

$$\begin{aligned}&\int_{-1}^{0}\mathrm{d}\mu\ \mathrm{Tr}\,\tilde{R}(\mu) \\ &= \ln\left(\frac{T\sin(T\sqrt{2\lambda})[(\lambda_0+\lambda_1)f_1(\lambda_1) - (\lambda_0-\lambda_1)f_2(\lambda_1)](\lambda_1^2-\lambda_0^2)^{1/2}}{(2\lambda_1)^{3/2}\{\sin[T(\lambda_1-\lambda_0)^{1/2}]\sin[T(\lambda_1+\lambda_0)^{1/2}]\}f_1(\lambda_0)}\right).\end{aligned} \tag{3.41}$$

Inserting in (3.41) the explicit values

$$\begin{aligned}&\lambda_1+\lambda_0 = \Omega^2, \qquad \lambda_1-\lambda_0 = \omega^2, \\ &f_1(\lambda_0) = -\omega\tan(\tfrac{1}{2}\omega T), \qquad f_1(\lambda_1) = 0, \\ &f_2(\lambda_1) = -(\Omega^2+\omega^2)^{1/2}\tan[\tfrac{1}{2}T(\Omega^2+\omega^2)^{1/2}],\end{aligned} \tag{3.42}$$

we obtain

$$\int_{-1}^{0} \mathrm{d}\mu\ \mathrm{Tr}\tilde{R}(\mu) = \ln\left[\left(\frac{\omega^2}{\omega^2+\Omega^2}\right)\left(\frac{\Omega T}{\sin \Omega T}\right)\left(\frac{\sin^2[(\frac{1}{2}T(\Omega^2+\omega^2)^{1/2}]}{\sin^2(\frac{1}{2}\omega T)}\right)\right]. \tag{3.43}$$

Hence, the normalisation factor Q_1 is given by

$$\begin{aligned} Q_1 &= \exp\left(\int_{-1}^{0} \mathrm{d}\mu\ \mathrm{Tr}\ \tilde{R}(\mu)\right) \\ &= [\omega^2/(\omega^2+\Omega^2)](\Omega T/\sin \Omega T)\{\sin^2[\tfrac{1}{2}T(\Omega^2+\omega^2)^{1/2}]/\sin^2(\tfrac{1}{2}\omega T)\}. \end{aligned} \tag{3.44}$$

Combining the results of (2.42), (3.14), (3.17) and (3.44) we arrive at the exact analytical form of the propagator

$$\begin{aligned} K(x, T; x_0, 0) &= (m/2\pi \mathrm{i}\hbar T)^{1/2}(\sin(\tfrac{1}{2}\omega T)/\omega)(\nu/\sin(\tfrac{1}{2}\nu T)) \\ &\quad \times \exp\left\{\frac{\mathrm{i}}{2\hbar}\frac{m\Omega^2}{\nu^2}\left[\frac{\omega^2}{\Omega^2 T}+\tfrac{1}{2}\nu \cot(\tfrac{1}{2}\nu \mathrm{T})\right](\mathrm{x}-\mathrm{x}_0)^2\right\}, \end{aligned}$$

where

$$\nu = (\Omega^2+\omega^2)^{1/2}. \tag{3.45}$$

A particular case of our kernel is obtained by taking the limit as $\omega \to 0$. In this case

$$G(t, s) = \tfrac{1}{4}m\Omega^2.$$

which is precisely the kernel used by Bezak and since then has been the subject of several applications. Taking the limit of the expression (3.46) as $\omega \to 0$ we obtain the propagator for this case

$$\begin{aligned} K_0(x, T; x_0, 0) &= \left(\frac{m}{2\pi \mathrm{i}\hbar T}\right)^{1/2} \frac{\Omega T}{2\sin(\frac{1}{2}\Omega T)} \\ &\quad \times \exp\left[\frac{\mathrm{i}m\Omega}{4\hbar}(\cot(\tfrac{1}{2}\Omega T))(x-x_0)^2\right] \end{aligned} \tag{3.46}$$

which agrees with the result of Papadopoulos (1974) and others (Maheshwari 1975, Khandekar *et al* 1981, Dhara *et al* 1982). If we let $\Omega \to 0$ we recover from (3.45) the usual free particle propagator. Finally, if we use imaginary time $T = -\mathrm{i}\hbar\beta$ in equation (3.45), the resulting expression agrees with the recent result of Adamowski and Gerlach (1982).

4. Summary

The main contribution of this paper is the explicit evaluation of the propagator for the general non-local (two-time) action (1.1) within the polygonal path approach of Feynman. Our derivation emphasises that both the exponent and the normalisation factor in the propagator are determined once the solutions of certain integrodifferential equations are obtained. We have illustrated our technique for a known kernel $G(s, t)$. However, the general results derived in this paper imply the possibility of using in physical applications new kernels $G(s, t)$ for which the above equations have analytical solutions. Further, in many physical applications it may also be necessary to evaluate the propagator when the action contains an additional time-dependent force term. The analysis of this paper can be readily extended to cover this case too.

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